ON SOME NEW SOLUTIONS OBTAINABLE BY MEANS OF INVARIANT TRANSFORMATIONS

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With reference to the example of the equations of monoenergetic nonrelativistic beam of particles of like charge, it is shown how new noninvariant solutions can be obtained by means of invariant transformations (§ 1). The conditions under which Lorentz forces can be ignored and the electric field considered a potential field are obtained for nonstationary flows. Solutions that describe the passage through a plane diode of high-frequency current from the emitter in a high-frequency electric field for an arbitrary relationship between the constant component of the collector potential and the amplitude of the ac voltage across it are derived (§2). Multivelocity (the velocity vector is a multivalued function) beams, and also electrostatic beams that can be described by Vlasov's equations are examined (§3).

Given a system of differential equations (S) for $m \geq 1$ unknown functions u^k $(k = 1, \ldots, m)$ of $n - m \geq 1$ independent variables x^i $(i = 1, \ldots, n - m)$. The set of values (x, u) is considered as the set of coordinates of a point in n-dimensional space E_n. Any solution of this system $u = u(x)$ defines some manifold in E_n. All possible solutions of (S) specify in E_n some set M. Any invariant transformation of system (S) has the property that it does not lead out of M. In a number of cases, this makes it possible to obtain new solutions by means of invariant transformations, no limitations being imposed on the solutions transformed. For a given system (S), all transformations that preserve (S) and form a continuous group, can be obtained by the method developed by L. V. Ovsyannikov [1–3]. Note that new solutions arise only when the principal group G of system (S) allows other than merely elementary transformations: magnifications, rotations, and translations are, as a rule, useless.*

Below, solutions of the equations of a monoenergetic nonrelativistic beam of particles of like charge are examined as an example [6–8].

1. EXAMPLES OF NEW NONINVARIANT SOLUTIONS

In [8] it was shown that, besides a number of elementary transformations, the equations of a nonstationary beam in the absence of an external magnetic field admit the following independent transformations:

$$t' = t,\quad x' = x + \alpha f(t),\ y' = y,\ z' = z,\ u' = u + \alpha f'(t),$$
$$v' = v,\ w' = w \quad \varphi' = \varphi + \alpha f''(t)x, \qquad \rho' = \rho,$$
$$t' = t, \quad x' = x,\ y' = y + \beta g(t),\ z' = z,\ u' = u,$$
$$v' = v + \beta g'(t),\ w' = w \quad \varphi' = \varphi + \beta g''(t)y, \qquad \rho' = \rho,$$
$$t' = t,\ x' = x,\ y' = y,\ z' = z + \gamma h(t),\ u' = u,\ v' = v,$$
$$w' = w + \gamma h'(t) \quad \varphi' = \varphi + \gamma h''(t)z,\ \ \rho' = \rho. \quad (1.1)$$

* For example, the new solutions of the equations of motion of a compressible inviscid fluid obtained by A. A. Nikol'skii [4–5] are linked with the presence of a discrete group of transformations

$$\begin{array}{lllll} t' = -1/t, & \mathbf{r} = \mathbf{r}/t, & \mathbf{V}' = t\mathbf{V} - \mathbf{r}, & \rho' = t^3\rho, & p' = t^3 p \\ t' = -1/t, & \mathbf{r} = -\mathbf{r}/t, & \mathbf{V}' = -t\mathbf{V} + \mathbf{r}, & \rho' = -t^3\rho, & p' = -t^3 p \\ t' = t & \mathbf{r} = -\mathbf{r}, & \mathbf{V}' = -\mathbf{V}, & \rho' = -\rho, & p' = -p \\ t' = t & \mathbf{r} = \mathbf{r}, & \mathbf{V}' = \mathbf{V}, & \rho' = \rho, & p' = p \\ & \mathbf{r} = \{x, y, z\}, & \mathbf{V} = \{u, v, w\}, & \varkappa = 5/3 & \end{array}$$

formed by the elements of the continous group found in [3].

Here, t is time; x, y, z are Cartesian coordinates; u, v, w are the velocity components in these coordinates; φ is the scalar potential; ρ is the space-charge density; f, g, h are arbitrary functions of time; α, β, Υ are the parameters of the continuous groups of transformations; the dimensionless variables used in [6-8] are again employed.

Transformations (1.1) can be represented as a single formula

$$t' = t,\quad x' = x + f(t),\quad y' = y + g(t),\quad z' = z + h(t)$$
$$u' = u + f'(t),\quad v' = v + g'(t),\quad w' = w + h'(t)$$
$$\varphi' = \varphi + f''(t)x + g''(t)y + h''(t)z,\quad \rho' = \rho. \quad (1.2)$$

It was shown in [6, 7] that all known solutions of the equations of a stationary beam are invariant solutions, with the exception of a few solutions that do not satisfy the conditions of thermionic emission [9-14]. Of these five noninvariant solutions, three are electrostatic [9-11]. The system of equations of a regular [15] beam can be reduced to a single nonlinear fourth-order differential equation in W—the action relative to the particle mass [16].

1°. In [9] Meltzer described plane flow along hyperbolic trajectories with constant space-charge density

$$u = ax,\quad v = -ay,\quad \varphi = {}^1/_2 a^2 R^2,$$
$$\rho = 2a^2,\quad W = {}^1/_2 a(x^2 - y^2), \qquad (1.3)$$
$$u = ay,\quad v = bx,\qquad \varphi = {}^1/_2 ab R^2,$$
$$\rho = 2ab \quad (R^2 = x^2 + y^2). \qquad (1.4)$$

The first of these flows will be regular. Using transformation (1.2), we obtain the nonstationary solution corresponding to (1.3)

$$u = a[x + f(t)] - f'(t),$$
$$v = -a[y + g(t)] - g'(t),\qquad \rho = 2a^2.$$
$$\varphi = {}^1/_2 a^2R^2 + (a^2 f - f'')x + (a^2 g - g'')y. \qquad (1.5)$$

At the initial moment t = 0, at $x_0(0) = y_0(0) = 0$, the circles R = const will be equipotential curves. Subsequently, this family of curves is displaced according to the law

$$x_0 = {}^1/_4 a^2(f'' - a^2 f),\qquad y_0 = {}^1/_4 a^2(g'' - a^2 g).$$

Here, x_0, y_0 are the coordinates of the center of the family of circles. Because of the arbitrariness of the functions f(t) and g(t), the particle trajectories may also be arbitrary. It is apparent that

$x_0 = x_0(t) \equiv 0,\ y_0 = y_0(t) \equiv 0$, if

$$f(t) = \alpha\,\mathrm{ch}\,at + \beta\,\mathrm{sh}\,at, \qquad g(t) = \gamma\,\mathrm{ch}\,at + \delta\,\mathrm{sh}\,at$$

($\alpha, \beta, \gamma, \delta$ are arbitrary constants).

In this case, the particle trajectories are given by the expressions

$$x = {}^1/_2(\beta - \alpha)e^{-at} + Ae^{at},$$

$$y = -{}^1/_2(\gamma + \delta)e^{at} + Be^{-at} \qquad (A, B = \mathrm{const}).$$

It is interesting that solution (1.3) is invariant relative to transformation (1.2) if $f \sim e^{at}$, $g \sim e^{-at}$.

The functions $f(t)$ and $g(t)$ can be selected so that the set of curves φ = const with center at x_0, y_0 executes a finite motion. For example, when $u = x + \sin t$, $v = -y + \sin t$, the center x_0, y_0 moves about the circle $x_0^2 + y_0^2 = 1/8$ with constant velocity and the trajectories will be nonmonotonic curves

$$x = -{}^1/_2(\sin t + \cos t) + Ae^t, \quad y = {}^1/_2(\sin t - \cos t) + Be^{-t}.$$

Similar results are easily obtained for (1.4), and also for the solution that extends (1.3) to the three-dimensional case [10].

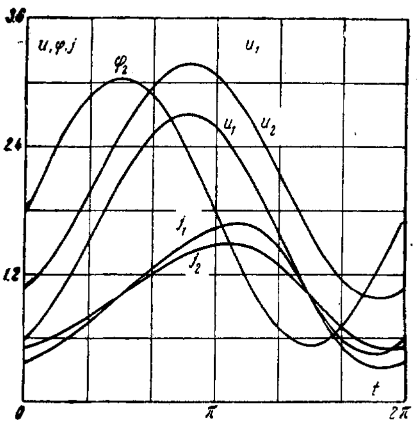

Fig. 1

2°. The plane periodic flow studied in [11] is defined by the formulas

$$W = \mathrm{Re}(-2ia \ln \mathrm{sc}\, z) = 2a\, \mathrm{arc\,tg}(\mathrm{tg}\, x\, \mathrm{th}\, y),$$

$$\cos 2x + \mathrm{ch}\, 2y = \mathrm{const} \qquad \varphi = \varphi_0 \frac{\mathrm{ch}\, 2y - \cos 2x}{\mathrm{ch}\, 2y + \cos 2x},$$

$$\rho = \frac{16\varphi_0}{(\mathrm{ch}\, 2y + \cos 2x)^2} \qquad (\varphi_0 = \mathrm{const}). \qquad (1.6)$$

The second equation in (1.6) is the equation of the trajectories.

Note, first of all, that, besides (1.6), there is an analogous stationary solution with arbitrary period. It can be obtained from (1.6) by an extension transformation with the infinitesmal operator

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2\rho\frac{\partial}{\partial \rho}$$

and is specified by the expression

$$\varphi = \varphi_0 \frac{\mathrm{ch}\, \alpha y - \cos \alpha x}{\mathrm{ch}\, \alpha y + \cos \alpha x}, \qquad \rho = \frac{4\alpha^2\varphi_0}{(\mathrm{ch}\, \alpha y + \cos \alpha x)^2},$$

$$\cos \alpha x + \mathrm{ch}\, \alpha y = \mathrm{const} \qquad (1.7)$$

If we apply transformation (1.2) to (1.7), we have

$$u = \frac{2a\, \mathrm{sh}\,[\alpha y + g(t)]}{\mathrm{ch}\,[\alpha y + g(t)] + \cos[\alpha x + f(t)]} - \frac{1}{\alpha} f'(t)$$

$$v = \frac{2a \sin[\alpha x + f(t)]}{\mathrm{ch}\,[\alpha y + g(t)] + \cos[\alpha x + f(t)]} - \frac{1}{\alpha} g'(t)$$

$$\varphi = \varphi_0 \frac{\mathrm{ch}\,[\alpha y + g(t)] - \cos[\alpha x + f(t)]}{\mathrm{ch}\,[\alpha y + g(t)] + \cos[\alpha x + f(t)]} - \frac{1}{\alpha}[f''(t)x + g''(t)y].$$

By selecting periodic functions as $f(t)$, we obtain a solution that is periodic not only in space but also in time.

2. NONSTATIONARY PROCESSES IN A PLANE DIODE

All the essentially different invariant solutions of a stationary beam were constructed in [6, 7]. When H = 0 each of these solutions can be subjected to transformation (1.2). As a result, a nonstationary image of the corresponding stationary flow is obtained. It should be noted that these solutions, which are similar to the H-solutions of the equations of a stationary beam, are not similar to any of the nonstationary invariant solutions constructed in [8], i.e., they cannot be obtained from them by means of transformations of the principal group G_t of equations of the nonstationary beam. In the study of nonstationary flows, therefore, the examination of such solutions is of interest. The solutions corresponding to one-dimensional flow between parallel planes admit a particularly simple interpretation [17–20].

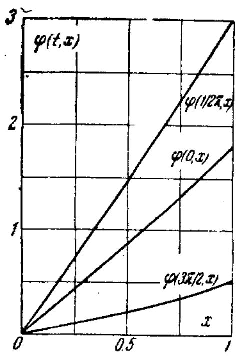

Fig. 2

Let us dwell in more detail on the solution obtained using transformation (1.2) from the Child-Langmuir solution [17, 18], which describes a plane diode under total space charge conditions.

Let us introduce the dimensionless variables t°, x°, u°, φ°, ρ°, j° defined by the formulas

$$t = \left(\frac{a}{18\pi\eta j_0}\right)^{1/3} t^\circ, \qquad \varphi = \left(\frac{9\pi j_0 a^2}{\sqrt{2\eta}}\right)^{2/3} \varphi^\circ, \qquad \rho = \left(\frac{j_0^2}{18\pi\eta a^2}\right)^{1/3} \rho^\circ,$$

$$x = a x^\circ, \qquad u = (18\pi\eta j_0 a^2)^{1/3} u^\circ, \qquad j = j_0 j^\circ \qquad (\eta = |e|/m). \quad (2.1)$$

The values that determine the Child-Langmuir solution have been selected as characteristic quantities: a is the interelectrode distance; j_0 is the emission-current density; the potential is referred to the collector potential that ensures a curren j_0 at distance a between the electrodes, etc.

In these variables, the Child-Langmuir solution has the form

$$\varphi^\circ = (x^\circ)^{4/3}, \qquad u^\circ = (x^\circ)^{2/3}, \qquad \rho^\circ = (x^\circ)^{-2/3}, \qquad j^\circ = 1. \quad (2.2)$$

Omitting the dimensionless-value symbol and applying transformation (1.2) to (2.2), we have

$$u = [x + f(t)]^{2/3} - f'(t), \quad \varphi = [x + f(t)]^{4/3} - f''(t)\,x - [f(t)]^{4/3},$$

$$\rho = [x + f(t)]^{-2/3}, \qquad j = 1 - f'(t)\,[x + f(t)]^{-2/3}. \quad (2.3)$$

The obtained solution (2.3) describes certain processes in a plane diode $0 \le x \le 1$. The function $f(t)$ can be selected so that the collector potential

$$\varphi_2 = (1 + f)^{4/3} - f^{4/3} - f'' \quad (2.4)$$

is a periodic function of time (subscripts 1 and 2 refer to emitter and collector, respectively). It should be noted that $f(t)$ is arbitrary with accuracy to the correctness of the initial equations [8]. With the intention of examining rapidly oscillating solutions of the form of (2.3), we obtain bounds that define the domain of applicability of the equations of a nonstationary beam used in [8].

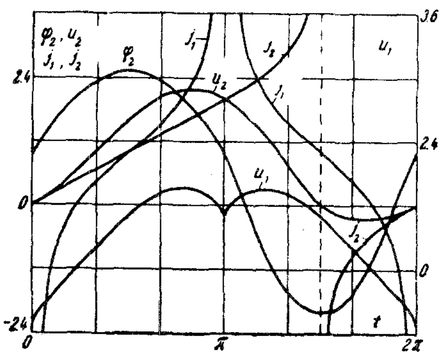

Fig. 3

Let the electric field on the collector of the plane diode be given by the expression

$$E_x(a) = E_0 + A \sin\omega t.$$

The characteristic value of the space-charge density ρ_* is found from the equation

$$\operatorname{div} \mathbf{E} = 4\pi\rho, \qquad \rho_* = (E_0 + A) / 4\pi a.$$

If it is considered that, in the case in question, displacement currents play no less a role than convection currents and that the aim is to obtain conditions under which the flow is close to one-dimensional, then given one more Maxwell equation we can determine the characteristic value of the magnetic-field strength H_*

$$\operatorname{rot} \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\rho\mathbf{V}, \qquad H_* = \frac{\omega b}{c} A = (E_0 + A)\frac{V_*}{c}\frac{b}{a}.$$

Here, b is the transverse dimension of the beam. The equations of motion of a charged particle

$$\frac{d\mathbf{V}}{dt} = \eta\mathbf{E} + \frac{\eta}{c}\mathbf{V} \times \mathbf{H}$$

make it possible to establish conditions under which the Lorentz forces are negligible as compared with the electric field forces

$$\frac{\omega b V_*}{c^2} A \ll E_0 + A, \qquad \left(\frac{V_*}{c}\right)^2 \frac{b}{a} \ll 1.$$

Hence, we obtain the limitations on the frequency ω and the relative beam dimensions b/a

$$\omega \ll \frac{c^2}{b V_*}\left(\frac{E_0}{A} + 1\right), \qquad \frac{b}{a} \ll \left(\frac{c}{V_*}\right)^2. \quad (2.5)$$

The second condition of (2.5) agrees (with accuracy to a constant factor that is not significant in order of magnitude under the bounds) with the condition established in [21, 22] for stationary nonrelativistic flows.

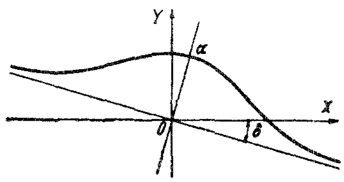

Fig. 4

For an electron beam at b ~1 cm, V_* ~10^9 cm/sec, E_0 ~A, j_0 ~10 ma/cm^2, we find that the frequency and longitudinal dimensions must satisfy the inequalities

$$\omega \ll 10^{12}\ \text{Hz} \qquad a \gg 10^{-3}\ \text{cm}$$

In dimensionless variables (2.1), we have $\omega^* \ll 10^3$. The dimensionless frequency ω^* ~1 corresponds to ω ~10^9 cps.

Thus, when inequalities (2.5) are fulfilled, it may be assumed that $\mathbf{E} = -\nabla\varphi$.

Figure 1 shows how u, φ, and j vary with time when $f = 2 + \sin t$ at emitter and collector ($\omega^* = 1$). In this case function (2.4) is approximated by the expression $\varphi_a = 1.24 \sin t + 1.8$. In the first half-cycle, φ_a and (2.4) agree, for all practical purposes; in the second half-cycle, the maximum difference in amplitude is not over 3%. Of course, when $\varphi_2 = \alpha \sin t + \beta$ and for any given $\varphi_2(t)$, the problem can be solved accurately, but this requires numerical integration of Eq. (2.4), or a more complicated equation, if we start with a stationary solution with arbitrary conditions on the emitter [19, 20]. Figure 2 shows the potential distribution at certain fixed moments in time. When $f = \sin t$, the right and left planes emit alternately (Fig. 3). At the ends and in the middle of the interval $0 \leq t \leq 2\pi$, $\partial u_1/\partial t$, not the total velocity derivative, increases without bound. The fact that the convection-current density goes to infinity at certain moments of time does not spoil the solution, since the current from a finite surface remains finite over a finite time interval

$$J = \int_0^t j d\xi = \int_0^t [1 - \cos \xi (x + \sin \xi)^{-2/3}] \, d\xi =$$

$$= t + 3 [x^{1/3} - (x + \sin t)^{1/3}] .$$

A solution of the form of (2.3) can describe the oscillatory conditions in a plane diode with nonuniform space-charge distribution, which cannot be obtained by construction of invariant nonstationary flows [8].

Note that an analytic solution of the problem of the behavior of a single electron in a high-frequency field is known [23]. The equations of a radio-frequency diode under total space charge conditions were numerically integrated in [24].

Nonstationary analytic solutions that correspond to annular electrostatic beams can be written without difficulty [25, 26]. However, they do not permit such a simple interpretation as do the flows examined above. For example, for the transform

$$\varphi = -{}^1/_2 [(x + f)^2 + (y + g)^2]^{-1} - [f'' (x + f) + g'' (y + g)]$$

the zero equipotential at the initial moment of time when $f(0) = f''(0) = g(0) = g''(0) = 0$ will be a circle of infinitely large radius, but subsequently in the noninertial frame of reference $X = x + f(t)$, $Y = y + g(t)$, it has the form shown in Fig. 4 and is defined by the formulas

$$R = \varkappa [\csc (\psi + \delta)]^{1/3}, \quad \varkappa = [4 (f''^2 + g''^2)]^{-1/6}, \quad \operatorname{tg} \delta = f'' / g''.$$

Here, R, ψ are polar coordinates in the system X, Y. Motion of the equipotential surfaces is accompanied by their deformation.

§3. MULTIVELOCITY BEAMS AND BEAMS DEFINED BY VLASOV'S EQUATIONS

Above we examined solutions with a single-valued velocity vector. A multivelocity beam of charged particles with the same value and sign of the specific charge η, for which $\mathbf{V}$ is an s-valued function, is formed by a finite number s of elementary monoenergetic beams. It is therefore described by a system of equations of a monoenergetic beam [6–8], the only difference being that the equation of current conservation and the boundary conditions must be written out for each of the s beams, while $\Sigma \rho_{(s)}$, where $\rho_{(s)}$ is the density of the s-th elementary beam, should be used as the space-charge density ρ in determining the potential.

From the above, it follows that the equations of a multivelocity beam admit transormations (1.2). By using these, we can construct some nonstationary transform in accordance with any multivelocity stationary solution.

It is known [19] that in a plane diode with emission from both electrodes (current densities j_1 and j_2, respectively) when the velocity is two-valued only with respect to direction, it is possible to get the same sets of conditions as in the case of emission from one plane with current density $j_0 = j_1 + j_2$. It is therefore not difficult to construct a nonstationary solution with high-frequency currents from both planes that corresponds to (2.3). In the general case, when the velocity is two-valued with respect to magnitude as well as direction, the solution of the equations of the beam is expressed in terms of elliptic integrals. A nonstationary solution can be constructed in this case, too.

Transformations of the form of (1.2) preserve the equations of a multivelocity beam with any, arbitrarily large numbers of elementary monoenergetic beams. It is natural to expect that this property is also preserved when $s \to \infty$, i.e., on going over to a description by means of the distribution function $F(\mathbf{r}, \mathbf{V})$. It is easy to see that in the absence of a magnetic field, transformations (1.2) leave Vlasov's equations invariant. By means of these transformations, we can find, for example, a nonstationary solution with a distribution function at the emitter given by the expression

$$F_1 = F_0 \exp \left\{ - \frac{m [u_1 - f'(t)]^2}{2kT_1} \right\} \qquad (F_0 = \text{const})$$

which corresponds to the solution for a plane diode [27–30] with a Maxwellian velocity distribution of the emitted particles $F_1 = F_0 \exp(-mu_1^2 / 2kT_1)$.

REFERENCES

1. L. V. Ovsyannikov, "Groups and invariant-group solutions of differential equations," DAN SSSR, vol. 118, no. 3, 1958.

2. L. V. Ovsyannikov, "The group properties of the equation of nonlinear heat conduction," DAN SSSR, vol. 125, no. 3, 1959.

3. L. V. Ovsyannikov, "The group properties of differential equations [in Russian], Izd. SO AN SSSR, Novosibirsk, 1962.

4. A. A. Nikol'skii, "Invariant transformation of equations of motion of an ideal monatomic gas and new classes of exact solutions," PMM, vol. 27, no. 3, 1963.

5. A. A. Nikol'skii, "Invariant transformations of the equations of motion of an ideal gas for special cases," Inzh. zh., vol. 3, no. 1, 1963.

6. V. A. Syrovoi, "Invariant-group solutions of the equations of a one-dimensional stationary charged-particle beam," PMTF, no. 4, 1962.

7. V. A. Syrovoi, "Invariant-group solutions of the equations of a three-dimensional stationary charged-particle beam," PMTF, no. 3, 1963.

8. V. A. Syrovoi, "Invariant-group solutions of the equations of a nonstationary charged-particle beam," PMTF, no. 1, 1964.

9. B. Meltzer, "Electron flow in curved paths under space-charge conditions," Proc. Phys. Soc. B, vol. 62, no. 355, 1949.

10. B. Meltzer, "Electron flow in curved paths under space-charge conditions," Proc. Phys. Soc. B, vol. 62, no. 360, 1949.

11. P. T. Kirstein, "The complex formulation of the equations of two-dimensional space-charge flow," J. Electr. Contr., vol. 4, no. 5, 1958.

12. G. Kent, "Generalized brillouin flow," Communic. Electr., vol. 79, no. 48, 1960.

13. L. R. Walker, "Generalization of brillouin flow," J. Appl. Phys., vol. 26, no. 6, 1955.

14. K. Pöschl and W. Veith, "Generalized brillouin flow," J. Appl. Phys., vol. 33, no. 3, 1962.

15. D. Gabor, "Dynamics of electron beams," Proc. IRE, vol. 33, no. 11, 1945.

16. K. Spangenberg, "Use of the action function to obtain the general differential equations of space charge flow in more than one dimension," J. Franklin Inst., vol. 232, no. 4, 1941.

17. C. D. Child, "Discharge from hot CaO," Phys. Rev., vol. 32, no. 5, 1911.

18. I. Langmuir, "The effect of space charge and residual gases on thermionic currents in high vacuum," Phys. Rev., vol. 3, no. 5, 1913.

19. C. E. Fay, A. L. Samuel, and W. Shockley, "On the theory of space charge between parallel plane electrodes," Bell. System Techn. J., vol. 17, no. 1, 1938.

20. H. F. Ivey, "Cathode field in diodes under partial space-charge conditions," Phys. Rev., vol. 76, no. 4, 1949.

21. B. Meltzer, "Magnetic constriction in simple diodes," Nature, vol. 181, no. 4619, 1958.

22. B. Meltzer, "Magnetic forces and relativistic speeds in stationary electron beams," J. Electr. Contr., vol. 4, no. 4, 1958.

23. W. Sackinger, "Electron streams in an oscillating electric field," J. Appl. Phys., vol. 33, no. 5, 1962.

24. K. W. Hinkel, "Über den raumladungsbegrenzten Influenzstrom in einer ebenen Diode bei grossen Signalen und langen Elektronenlaufzeitwinkeln unter Berücksichtigung von Sekundäremission der Elektronen," Microwaves. Proc. 4th Int. Congr. Microwave Tubes. Centr. Publ Comp., Eindhoven, 1963.

25. B. Meltzer, "Single-component stationary electron flow under space charge conditions," J. Electr., vol. 2, no. 2, 1956.

26. W. M. Mueller, "Necessary and sufficient trajectory conditions for dense electron beams," J. Electr. Contr., vol. 5, no. 6, 1959.

27. I. Langmuir, "The effect of space-charge and initial velocities on the potential distribution between parallel plane electrodes," Phys. Rev., vol. 21, no. 4, 1923.

28. F. H. Reynolds, "The potential distribution and thermionic current between parallel plane emitters," Proc. IRE, vol. 108, no. 13, part C, 1961.

29. P. A. Lindsay and F. W. Parker, "Potential distribution between two plane emitting electrodes," J. Electr. Contr., vol. 7, no. 4, 1959.

30. P. A. Lindsay and F. W. Parker, "Potential distribution between two plane emitting electrodes II. thermionic engines," J. Electr. Contr., vol. 9, no. 2, 1960.

5 March 1964 Moscow